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ON CERTAIN SPACE AND SURFACE INTEGRALS.*

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IN the proceedings of the American Academy of Arts and Sciences, May, 1891, Professor B. O. Peirce has given what may be considered as a remarkable extension of a well-known formula of the integral calculus. The formula is †

$$\int\int \frac{du}{dx}\,dxdy = \int u \cos a\, ds\,,$$

the integration in the second member being taken around a closed curve, and that in the first member over the included area, while a denotes the inolination of the exterior normal at any point of the curve to the axis of x. The enunciation of the extension may be given as follows. Let the points of a plane be referred to a system of orthogonal curvilinear co-ordinates $\xi = f_1(x, y)$, $\eta = f_2(x, y)$. If within a closed curve T, the function V and its first space derivatives are finite, continuous, and single-valued, and if the auxiliary quantities h_1, h_2, given by the equations

$$h_1^2 = (D_x\xi)^2 + (D_y\xi)^2,$$
$$h_2^2 = (D_x\eta)^2 + (D_y\eta)^2,$$

are always greater than zero, we have

$$\int\int h_1 h_2 \frac{d}{d\xi}\left[\frac{V}{h_2}\right] d\sigma = \int V \cos a\, ds\,,$$

where the single integration is along the curve T, and the double integration over the included area, while a denotes the inclination of the exterior normal at any point of the curve to the line of constant η at the same point.

Professor Peirce has indicated elsewhere ‡ that a similar result holds in tridimensional co-ordinates.

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* Read before the American Association for the Advancement of Science, August, 1891.

† Byerly's Integral Calculus, 2d edition, p. 196.

‡ Educational Times, February, 1891.

To obtain this, let us suppose that ρ_1, ρ_2, ρ_3 denote orthogonal co-ordinates, that h_1, h_2, h_3 have their usual significance *

$$h_1^2 = \left[\frac{d\rho_1}{dx}\right]^2 + \left[\frac{d\rho_1}{dy}\right]^2 + \left[\frac{d\rho_1}{dz}\right]^2,$$

$$h_2^2 = \left[\frac{d\rho_2}{dx}\right]^2 + \left[\frac{d\rho_2}{dy}\right]^2 + \left[\frac{d\rho_2}{dz}\right]^2,$$

$$h_3^2 = \left[\frac{d\rho_3}{dx}\right]^2 + \left[\frac{d\rho_3}{dy}\right]^2 + \left[\frac{d\rho_3}{dz}\right]^2,$$

and are all greater than zero, and that α is the inclination of the exterior normal at any point of a closed surface T to the line of constant ρ_2 and ρ_3 at the same point. It is to be observed that, when this line passes from without to within the closed surface, α is obtusé, and in the opposite case, acute. We have then

$$\int\int\int h_1 h_2 h_3 \frac{d}{d\rho_1}\left[\frac{W}{h_2 h_3}\right] dv = \int\int W \cos\alpha \, d\sigma .$$

For integrating the first member with respect to ρ_1, we obtain

$$\int\int\int \frac{d}{d\rho_1}\left[\frac{W}{h_2 h_3}\right] d\rho_1 \, d\rho_2 \, d\rho_3 = \int\int \Sigma \pm W \frac{d\rho_2 d\rho_3}{h_2 h_3}$$
$$= \int\int W \cos\alpha \, d\sigma ,$$

the last integral following since the infinitely small square $\frac{d\rho_2}{h_2}\frac{d\rho_3}{h_3}$ may be regarded as the projection of $d\sigma$ upon the surface ρ_1. Two other formulas of the same form are obtained by permuting the subscripts.

The results that may be obtained by applying the preceding to polar and cylindrical co-ordinates are of interest. Calculating the auxiliary quantities h by means of the equations of transformation from polar to rectangular co-ordinates, we have

$$h_1 = 1 , \quad h_2 = \frac{1}{r \sin\theta} , \quad h_3 = \frac{1}{r} ;$$

whence

$$\int\int\int \frac{1}{r^2}\frac{d}{dr}(r^2 W)\, dv = \int\int W \cos\alpha \, d\sigma ,$$

$$\int\int\int \frac{1}{r\sin\theta}\frac{d}{d\varphi} W dv = \int\int W \cos\beta \, d\sigma ,$$

$$\int\int\int \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta \,.\, W)\, dv = \int\int W \cos\gamma \, d\sigma ;$$

* Todhunter's Functions, p. 220.

α, β, γ being the inclinations of the exterior normal to the lines of constant φ and θ, θ and ρ, ρ and φ, respectively.

In the case of cylindrical co-ordinates

$$h_1 = 1\,, \quad h_2 = \frac{1}{r}\,, \quad h_3 = 1\,.$$

Hence

$$\iiint \frac{1}{r}\frac{d}{dr}(r\,W)\,dv = \iint W\cos\alpha\,d\sigma\,,$$

$$\iiint \frac{1}{r}\frac{d}{d\theta}\,W\,dv = \iint W\cos\beta\,d\sigma\,,$$

$$\iiint \frac{d}{dz}\,W\,dv = \iint W\cos\gamma\,d\sigma\,,$$

α, β, γ being the inclinations of the exterior normal to the lines of constant θ and z, z and r, r and θ, respectively. The last of these formulas is identical with the corresponding one in rectangular co-ordinates.* The preceding two are similar in form to those in plane polar co-ordinates.†

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* Byerly's Integral Calculus, p. 200.

† Cf. B. O. Peirce, loc. cit.